

How fading affects CDMA: An asymptotic analysis with linear receivers

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Abstract

Using asymptotic analysis, we study the effect of frequency-flat fading on CDMA systems with linear receivers and random spreading sequences. Specifically, we let the number of users grow without bound while the ratio of number of users to spreading sequence length is kept fixed to a value α . We treat separately the cases of slow fading (nonergodic channel) and of fast fading (ergodic channel). For the former channel we derive the outage probability, while for the latter we compute the channel capacity. In both cases multiple classes of users with different qualities of service are dealt with. As $\alpha \rightarrow \infty$ the system throughput tends to the same limit of 1.44 bit/symbol as for the non-fading channel with both single-user matched filter (SUMF) and linear minimum mean-square-error (MMSE) receivers. The outage probability exhibits a floor for all α with the SUMF receiver, while with MMSE receiver the floor is present only for $\alpha \geq 1$. We also address the tradeoffs involved in the allocation of available bandwidth between spreading and coding.

Index terms—CDMA, fading channels, channel capacity, outage probability.

1 Introduction

We examine a synchronous CDMA single-cell system with error-control coding, operating on a channel affected by frequency-flat fading. The receiver consists of a linear front-end, viz., either a single-user matched filter (SUMF) or a linear minimum-mean-square error filter (MMSE) [1], followed by single-user decoding. The key performance measure here is the Signal-to-Interference plus Noise Ratio (SINR) at the output of the linear filter: users' quality of service can be expressed in terms of a target SINR [2]. We treat separately the cases of slow and fast fading, yielding non-ergodic and ergodic channels, respectively.

Our study is asymptotic, in the sense that the number of users grows without bound, while the ratio of number of users to spreading-sequence length is kept fixed to a given value α . The spreading sequences are random. After the pioneering work of [3, 4] (see also [5] and references therein), the asymptotic random-sequences approach emerged as a very powerful tool to characterize in many aspects the behavior of large CDMA systems where the assignment of spreading sequences to the users is pseudo-random (this is the case of the uplink of current CDMA systems like IS-95 or UMTS/IMT2000 [6]). Beyond its theoretical beauty, this method is useful since the performance of actual (finite dimensional) systems converges quickly to the infinite-dimensional asymptotics, which depend only on fundamental system parameters such as the *system load* (users per chip), the statistics of the *received* signal-to-noise ratio (SNR) and the constraints on the *transmit* power, thus making analysis independent on the system fine-tuning characteristics, like the assignment of spreading sequences.

Independent and parallel work on CDMA systems with fading can be found in [7, 8, 9]. In [7], the random-sequences asymptotic analysis is used to characterize the performance of linear receivers with linear MMSE data-aided channel estimation, both in flat and in multipath channels. The problem of optimal (centralized) power allocation maximizing the system throughput for an optimal joint detector is solved in [8]. Finally, [9] presents system throughput and outage probability analysis for linear and

optimal receivers, with and without power control, and derives independently the same power control strategy of [8] for the optimal detector.

In this paper we apply the approach of [3] to the case of linear receivers without power control. In the non-ergodic case, the SINR cumulative distribution function (cdf) yields immediately the outage probability, i.e., the probability that the actual SINR is below the SINR target. In the case of multiple classes of users, with different input power constraints, target SINRs and outage probability requirements, we find *system capacity region*, defined as the region of rates at which all user classes can meet their quality of service requirements (this region has been defined and studied in [3] in the case of non-fading AWGN channel, and should not be confused with the information theoretic capacity region of the multiple access channel [10]). In the ergodic case, performance is given in terms of system throughput. Assuming that all users transmit “Gaussian codes,” this is also determined by the SINR cdf [11]. In the case of multiple classes we find the ergodic system capacity region.

After a description of the system model (Section 2), we examine the outage probability of slow-fading channels (Section 3) and the capacity of fast-fading channels (Section 4). The coding-vs.-spreading tradeoff is finally addressed.

2 System model

We consider the uplink of a single-cell, synchronous DS-CDMA system. Our model involves K users and random spreading sequences of length L chips. As in [3, 4], we assume a large number of users ($K \rightarrow \infty$) and $K/L \rightarrow \alpha$ (a constant channel load as the length of the spreading sequences increases to accommodate the K users). Since the system is synchronous, sufficient statistics for (optimal) detection of all users is provided by a chip-matched filter sampled at the chip rate (we assume that the chip waveform satisfies the Nyquist criterion [12] for no inter-chip interference, so that the sequence of noise samples at the chip matched-filter output is i.i.d.). The received signal

L -chip column vector corresponding to one symbol interval is given by

$$\mathbf{y} = \sum_{k=1}^K c_k x_k \mathbf{s}_k + \mathbf{n} \quad (1)$$

where \mathbf{n} is a complex circularly-symmetric AWGN vector $\sim \mathcal{N}_c(\mathbf{0}, N_0 \mathbf{I})$, x_k is the complex modulation symbol of user k , \mathbf{s}_k is the spreading sequence of user k , made of binary antipodal chips $\pm 1/\sqrt{L}$ generated at random with uniform probability, and c_k is the frequency-flat complex fading gain, which includes the carrier phase shift of each user and remains constant over the time necessary to transmit a symbol. We assume that the base station receiver has perfect knowledge of all fading gains (the “channel-state information”) and that the demodulation is coherent.

User k is received with an instantaneous SNR $\gamma_k = z_k \Gamma_k$, where $z_k \triangleq |c_k|^2$ is the fading channel “power gain”, $\Gamma_k = \mathcal{E}_k/N_0$ is the *transmit* SNR and \mathcal{E}_k is the user transmit average energy per symbol. We assume that as $K \rightarrow \infty$ the empirical cdf of the received SNR’s, defined by

$$F_\gamma^{(K)}(x) \triangleq \frac{1}{K} \sum_{k=1}^K \mathbf{1}\{\gamma_k \leq x\}$$

($\mathbf{1}\{\mathcal{A}\}$ the indicator function of the event \mathcal{A}) converges almost everywhere to the cumulative distribution function $F_\gamma(x)$.

The receiver for user 1 (our reference user) is formed by a linear filter \mathbf{h}_1 producing the output $y_1 = \mathbf{h}_1^H \mathbf{y}$ followed by a single-user decoder operating on the sequence of filter output samples y_1 . We shall consider either the single-user matched filter (SUMF) $\mathbf{h}_1 = \mathbf{s}_1$ and the linear minimum mean-square error filter (MMSE)

$$\mathbf{h}_1 = \left[\sum_{k=1}^K z_k \mathcal{E}_k \mathbf{s}_k \mathbf{s}_k^H + N_0 \mathbf{I} \right]^{-1} \mathbf{s}_1$$

The SINR at the filter output is defined as

$$\beta_1 \triangleq \frac{\gamma_1 |\mathbf{h}_1^H \mathbf{s}_1|^2}{|\mathbf{h}_1|^2 + \sum_{k=2}^K \gamma_k |\mathbf{h}_1^H \mathbf{s}_k|^2}$$

Under the above assumptions, we have the following results [3]:

1. The SUMF output SINR converges in probability for large K to

$$\beta_1 = \frac{\gamma_1}{1 + \alpha \int_0^\infty x dF_\gamma(x)} \quad (2)$$

2. The MMSE output SINR converges in probability for large K to the (unique) real non-negative solution of the equation

$$\beta_1 = \frac{\gamma_1}{1 + \alpha \int_0^\infty \frac{x\gamma_1}{\gamma_1 + x\beta_1} dF_\gamma(x)} \quad (3)$$

2.1 Distribution of the output SINR

From (2), it is immediately apparent that the SINR β_1 is proportional to γ_1 . Similarly, since equation (3) depends only on the ratio β_1/γ_1 , this turns out to be a deterministic quantity. Thus, for both SUMF and MMSE receivers the SINR has, apart from a scale factor, the same probability distribution as the fading power gain.

2.2 More than one class of users

We may assume, following [3], that the users are partitioned into J classes, each class j being characterized by a transmit SNR Γ_j . We can think of the Γ_j 's as the transmit SNRs determined by some power-control mechanism, and of the fading z_k as some channel attenuation that the power control is not able to compensate, either because it is too fast (as for example with Rayleigh multipath) or because of inaccuracies in the power control loop (as for example with residual shadowing [13]). Each class has $p_j K$ users, where p_j is the fraction of users belonging to class j (obviously, $\sum_{j=1}^J p_j = 1$). Moreover, we assume that the fading gains z_k are i.i.d. (the fading statistics is the same for all users) and normalized so that $\int_0^\infty x dF_z(x) = 1$. With these assumptions it is immediate to see that

$$F_\gamma(x) = \sum_{j=1}^J p_j F_z(x/\Gamma_j)$$

where $F_z(x)$ is the fading-gain cdf. Let user 1 belong to class i . Because of the uncompensated fading, user-1 SINR is a random variable $\beta_{i,1}$. However, as shown in the

previous subsection, the ratio $\xi = \beta_{i,1}/(\Gamma_i z_1)$ is a non-random constant independent of i . From (2), for the SUMF we obtain

$$\xi^{(s)} = \frac{1}{1 + \alpha \sum_{j=1}^J p_j \Gamma_j} \quad (4)$$

From (3), for the MMSE we obtain $\xi = \xi^{(m)}$ as the solution of the equation

$$\xi = \frac{1}{1 + \alpha \sum_{j=1}^J p_j \Gamma_j \int_0^\infty \frac{x}{1 + x \xi \Gamma_j} dF_z(x)} \quad (5)$$

This solution is unique, real, and takes values in the interval $[0, 1]$. Moreover, if we rewrite (5) in the form $\xi = g(\xi)$, the iteration $\xi_{n+1} = g(\xi_n)$ converges to the solution for any initial value ξ_0 [3].

We conclude that any user k belonging to class i has (asymptotically) SINR $\beta_{i,k} = z_k \Gamma_i \xi$ (where $\xi = \xi^{(s)}$ or $\xi = \xi^{(m)}$ depending on the receiver employed). Then, the SINR cdf for all users is just given by the fading cdf after a scale change. For users of class i it is given by

$$F_i(x) = \mathbb{P}(\beta_{i,k} \leq x) = \mathbb{P}(z_k \leq x/(\Gamma_i \xi)) = F_z(x/(\Gamma_i \xi)) \quad (6)$$

Moreover, the SINRs of different users are (asymptotically as $K \rightarrow \infty$) statistically independent.

2.3 Methodological preamble

Here we list a number of points that describe the rationale behind the calculations that follow. We shall analyze a flat, slow fading channel for which the channel gain is constant for the whole duration of a code word, and a flat, fast fading channel for which the channel gain varies considerably during the transmission of a code word. The information-theoretic subtleties of dealing with fading channels are thoroughly described in [11]. Roughly speaking, we can think of a slow fading channel as a *compound channel*, i.e., as a collection of channels each of which is characterized by a fixed

set of power gains. An internal channel state process, independent of the input signals and of the noise, selects a particular channel in the compound and keeps this selection for the whole duration of a user codeword. The channel state is known at the receiver and unknown at the transmitters. Each channel in the compound has a well-defined capacity, but since the transmitters do not know the channel state realization, they might transmit at a rate above the capacity of the actually selected channel. This event is called *information outage*, and its probability is the *information outage probability*. In this setting, the compound channel capacity is not larger than the minimum of the capacities of the channels in the compound. If the infimum of the support of the fading probability distribution is 0 (i.e., if there is a non-zero probability that the channel gain is below any assigned positive threshold), the capacity of the slow fading channel is zero. In the case of fast fading, the channel gain experienced during the transmission of a code word varies sufficiently so that all fading realizations occurs with empirical probabilities arbitrarily close to their statistical probabilities. This ergodic behavior of fading makes channel capacity be equal to the channel mutual information *averaged* with respect to the fading statistics (and maximized with respect to the input probability distribution).

Flat, slow fading. The outage probability of the reference user is defined by

$$P_{\text{out}} \triangleq \mathbb{P}(\beta < \bar{\beta}) \quad (7)$$

The value of $\bar{\beta}$ may be chosen as follows [14]. For example, if we use a nonideal code with rate R bit/symbol achieving the target performance at a certain E_b/N_0 , we set $\bar{\beta} = RE_b/N_0$. If we consider instead an optimum code which operates at the Shannon limit for a Gaussian channel and we want a rate R , then we set

$$\bar{\beta} = 2^R - 1 \quad (8)$$

Flat, fast fading. We assume that all users generate their code book according to a complex circularly-symmetric Gaussian probability density function (pdf); hence,

the single-user channel seen at the output of any user's receiving filter is an additive Gaussian noise channel, whose capacity is

$$C = \mathbb{E}[\log_2(1 + \beta)] = \int_0^\infty \log_2(1 + x\xi\Gamma) dF_z(x) \quad (9)$$

System throughput. The *system throughput* η is defined as the total number of bit/symbol supported by the system. For large systems, it is possible to transmit close to one complex symbol per second per Hz, and hence to express η in bit/s/Hz. If all K users transmit at rate C , the system throughput is

$$\eta = \frac{KC}{L} = \alpha C \quad (10)$$

The value of the channel load that yields maximum throughput is defined as

$$\alpha_{\text{opt}} \triangleq \arg \max_{\alpha} \eta$$

3 Slow-fading channel

Outage probability for users of class i is defined as the probability that the SINR is below some threshold value $\bar{\beta}_i$ that depends on the coding scheme of class i . We obtain

$$P_{\text{out},i} = \mathbb{P}(\beta_{i,k} \leq \bar{\beta}_i) = F_z \left(\frac{\bar{\beta}_i}{\xi\Gamma_i} \right) \quad (11)$$

Assuming Gaussian codes and minimum distance decoding at the output of the receiving filter, each user can transmit with arbitrarily small error probability at rate $R_{i,k} = \log_2(1 + \beta_{i,k})$ [15], for sufficiently large code block lengths. Then, in the absence of further specification of class i user codes, it make sense to define the coding rate R_i bit/symbol for users in class i and choose the SINR threshold as $\bar{\beta}_i = 2^{R_i} - 1$.

3.1 System outage capacity

Let $\bar{\Gamma} = (\bar{\Gamma}_1, \dots, \bar{\Gamma}_J)$ be a vector of input SNR constraints, $\epsilon = (\epsilon_1, \dots, \epsilon_J)$ a vector of target outage probabilities, and $\mathbf{R} = (R_1, \dots, R_J)$ be a vector of coding rates. We would

like to find the set $\mathcal{R} \subseteq \mathbb{R}_+^J$ of rate vectors \mathbf{R} that can be assigned to the J classes such that, for all $i = 1, \dots, J$

$$P_{\text{out},i} \leq \epsilon_i \quad \text{and} \quad \Gamma_i \leq \bar{\Gamma}_i \quad (12)$$

We refer to the region \mathcal{R} as the *system outage capacity* region for the input and outage constraints specified by $\bar{\Gamma}$ and ϵ .

SUMF receiver. Define

$$\mu_i \triangleq \frac{2^{R_i} - 1}{\sup\{x \in \mathbb{R}_+ : F_z(x) = \epsilon_i\}} \quad (13)$$

(which reduces to $\mu_i \triangleq (2^{R_i} - 1)/F_z^{-1}(\epsilon_i)$ when $F_z(\cdot)$ is strictly monotonic and continuous) and, by using (11) and (4), rewrite the outage constraint as

$$\frac{\Gamma_i}{1 + \alpha \sum_{j=1}^J p_j \Gamma_j} \geq \mu_i$$

The maximum R_i is achieved when the above inequalities are satisfied with equality. The solution of the resulting system of equations has the form $\Gamma_i/\mu_i = \kappa$, where κ does not depend on i . Solving for κ we obtain

$$\kappa = \frac{1}{1 - \alpha \sum_{j=1}^J p_j \mu_j}$$

provided that

$$\alpha \sum_{j=1}^J p_j \mu_j \leq 1 \quad (14)$$

The resulting transmit SNR assignment (power control) is obtained from $\Gamma_i = \kappa \mu_i$ in the form

$$\Gamma_i = \frac{\mu_i}{1 - \alpha \sum_{j=1}^J p_j \mu_j}$$

By imposing the input constraint $\Gamma_i \leq \bar{\Gamma}_i$ for all $i = 1, \dots, J$, we obtain the desired final result

$$\alpha \sum_{j=1}^J p_j \mu_j \leq \min_{1 \leq i \leq J} \left\{ 1 - \frac{\mu_i}{\bar{\Gamma}_i} \right\} \quad (15)$$

MMSE receiver. By using (11) and (13), rewrite the outage constraint as

$$\Gamma_i \xi^{(m)} \geq \mu_i$$

Since $\xi^{(m)}$ does not depend on i , the solution for the above system (taken with equality) has the form $\Gamma_i/\mu_i = \kappa$, where $\kappa = 1/\xi^{(m)}$. By substituting in (5) and solving for κ we obtain

$$\kappa = \frac{1}{1 - \alpha \sum_{j=1}^J p_j \mu'_j},$$

where

$$\mu'_j \triangleq \int_0^\infty \frac{x \mu_j}{1 + x \mu_j} dF_z(x),$$

provided that

$$\alpha \sum_{j=1}^J p_j \mu'_j \leq 1 \tag{16}$$

The resulting transmit SNR assignment (power control) is obtained from $\Gamma_i = \kappa \mu_i$ in the form

$$\Gamma_i = \frac{\mu_i}{1 - \alpha \sum_{j=1}^J p_j \mu'_j}$$

By imposing the input constraint $\Gamma_i \leq \bar{\Gamma}_i$ for all $i = 1, \dots, J$, we obtain the desired final result

$$\alpha \sum_{j=1}^J p_j \mu'_j \leq \min_{1 \leq i \leq J} \left\{ 1 - \frac{\mu_i}{\bar{\Gamma}_i} \right\} \tag{17}$$

Effective bandwidth. Equations (15) and (17) are the generalization to the case of slow (non-ergodic) fading of the system capacity equations found in [3] in the case of no fading. Interestingly, the outage constraint and the presence of fading yield formally the same constraints for the fractions αp_i of class i users per system degree of freedom. In analogy with [3], we define the effective bandwidth B_i of class i users as the amount of degrees of freedom consumed in order to support rate R_i with outage probability ϵ_i .

By rewriting (14) in the form

$$\sum_{j=1}^J (p_j K) \mu_j \leq L$$

for the SUMF receiver we have

$$B_i^{(s)} = \mu_i$$

Similarly, by rewriting (16) in the form

$$\sum_{j=1}^J (p_j K) \mu'_j \leq L$$

for the MMSE receiver we have

$$B_i^{(m)} = \mu'_j$$

Obviously, $B_i^{(s)} \geq B_i^{(m)}$ and $B_i^{(m)} \leq 1$. For very high quality of service (large rates and/or small outage probabilities) μ_i becomes large. We observe that, with SUMF receiver, class- i users may require an unbounded number of degrees of freedom, while with MMSE receiver they will require at most 1 degree of freedom, as in the absence of fading [3].

3.2 Outage probability floor and near-far resistance

Consider the case of a single class, and neglect for simplicity the class index. The outage probability is given by $P_{\text{out}} = F_z(\bar{\beta}/(\xi\Gamma))$. As $\Gamma \rightarrow \infty$, for the SUMF receiver we have

$$\lim_{\Gamma \rightarrow \infty} \xi^{(s)} \Gamma = 1/\alpha$$

Therefore, the outage probability has a floor at $F_z(\alpha\bar{\beta})$ for all $\alpha > 0$. This is a consequence of the fact that CDMA with SUMF reception is interference-limited, and because of fading, there is always a non-zero probability that some interferer is so strong that drives the SINR below the target threshold.

With an MMSE receiver, if we let $\Gamma \rightarrow \infty$ in (5) we obtain the equation

$$\frac{\xi^{(m)}}{\xi^{(m)} + \alpha} = \xi^{(m)}$$

This has a positive solution for $\xi^{(m)}$ if and only if $\alpha < 1$ (recall that $0 \leq \xi^{(m)} \leq 1$), otherwise $\xi^{(m)} = 0$ is the only solution. We conclude that, if $\alpha < 1$ then $\lim_{\Gamma \rightarrow \infty} P_{\text{out}} = 0$ (no outage probability floor), otherwise there is an error floor. This is a consequence of the fact that the MMSE receiver is near-far resistant (i.e., not interference-limited) if the user spreading sequences are linearly independent, and that with sufficiently long random sequences linear independence is achieved with arbitrarily large probability if $\alpha < 1$ and with arbitrarily small probability if $\alpha > 1$ [1].

Example 1. Assume for the fading gains a log-normal distribution with log-standard deviation σ dB (the “shadowing factor”) and mean value $\mathbb{E}[z] = 1$. Letting $\xi^{(s)} \triangleq 1/(\Gamma^{-1} + \alpha)$, for the SUMF we get

$$P_{\text{out}}^s = 1 - Q \left[\sigma^{-1} \left(\mu_z + 10 \log_{10}[(2^R - 1)/\xi^{(s)}] \right) \right]$$

where $Q(x) \triangleq \mathbb{P}(\mathcal{N}(0, 1) > x)$. For the MMSE we obtain the equation

$$z_1/\beta = \Gamma^{-1} + \alpha \mathbb{E}_z \left[\frac{z}{z_1/\beta + z} \right] (z_1/\beta) \quad (18)$$

which can be solved iteratively through the recursion

$$\begin{aligned} \xi_n &= \Gamma^{-1} + \alpha \mathbb{E} \left[\frac{z}{\xi_{n-1} + z} \right] \xi_{n-1} \\ &= \Gamma^{-1} + \alpha \mathbb{E}_X \left[\frac{10^{\sigma X/10}}{\xi_{n-1} 10^{\mu_z/10} + 10^{\sigma X/10}} \right] \xi_{n-1} \\ &= \Gamma^{-1} + \frac{\alpha}{\sqrt{2\pi}} \xi_{n-1} \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\xi_{n-1} 10^{(\mu_z - \sigma x)/10} + 1} dx \end{aligned}$$

where $X \sim \mathcal{N}(0, 1)$, with the initial value $\xi_0 = 1$. This iteration converges to $\lim_{n \rightarrow \infty} \xi_n = z_1/\beta$. The computation of the outage probability reduces to determining the cumulative distributionfunction of a log-normal variates:

$$P_{\text{out}}^s = 1 - Q \left[\sigma^{-1} \left(\mu_z + 10 \log_{10}[(2^R - 1)/\xi^{(m)}] \right) \right]$$

With the SUMF receiver, since $\lim_{\Gamma \rightarrow \infty} \xi^{(s)} = \alpha^{-1}$, the outage probability floor is

$$1 - Q \left[\sigma^{-1} \left(\mu_z + 10 \log_{10}[(2^R - 1)\alpha] \right) \right] \quad (19)$$

for any value of α .

The outage probability, as well as its floor, is illustrated in Fig. 1, obtained by plotting P_{out} versus E_b/N_0 with the MMSE and SUMF receivers, $\alpha = 0.2, 0.5, 0.8, 1, 1.2, 1.5$, $\bar{\beta} = 2^R - 1$ with rates $R = 1$ and 2 bit/symbol, and log-normal fading with shadowing factor [16] $\sigma = 2$ and 8 dB. The outage probability degrades, as expected, by increasing either R , α , or σ , which represent the user rate, system load, and shadowing level, respectively.

Fig. 2 shows the outage capacity region with SUMF and MMSE receiver in a system with two user groups: the first one has 90% of the users transmitting with an outage probability 0.1 and a signal-to-noise ratio of $\bar{\Gamma}_1 = 10$ dB; the second one has 10% of the users transmitting with an outage probability 0.01 and a signal-to-noise ratio of $\bar{\Gamma}_2 = 13$ dB. The channel statistics are log-normal with log-standard deviation $\sigma = 2$ dB, and $\alpha = 0.2, 0.5, 1$, and 2.

4 Ergodic fading

Consider again J classes of users. From (9), the rate at which a user in class i can communicate reliably in an ergodic-fading regime is given by

$$R_i = \int_0^\infty \log_2(1 + x\xi\Gamma_i) dF_z(x) \quad (20)$$

where $\xi = \xi^{(s)}$ or $\xi = \xi^{(m)}$ depending on the linear receiver employed. We want to determine the set of rates $\mathbf{R} = (R_1, \dots, R_J)$ achievable by the system with input constraint $\Gamma \leq \bar{\Gamma}$, load α , and fractions p_1, \dots, p_J of users belonging to classes $1, \dots, J$.

Let $f(y) \triangleq \int_0^\infty \log_2(1 + xy) dF_z(x)$. This function is monotonically increasing for $y \geq 0$. Next, for all $i = 1, \dots, J$, define

$$\nu_i \triangleq f^{-1}(R_i)$$

The solution of the rate equations (20) with respect to the Γ_i 's has the form

$$\xi\Gamma_i = \nu_i$$

i.e., $\Gamma_i/\nu_i = 1/\xi$, a constant independent of i .

We can now see that the problem we are dealing with here is formally identical to that solved in previous Section and leading to the outage capacity region: thus, power control, capacity, and effective bandwidth formulas in the ergodic case can be obtained simply by replacing ν_j for μ_j in the results for the outage capacity. Explicitly, for the SUMF receiver we have

$$\xi = 1 - \alpha \sum_{j=1}^J p_j \nu_j$$

provided that

$$\alpha \sum_{j=1}^J p_j \nu_j \leq 1$$

The power-control equation is

$$\Gamma_i = \frac{\nu_i}{1 - \alpha \sum_{j=1}^J p_j \nu_j}$$

and, by imposing the input constraint $\Gamma_i \leq \bar{\Gamma}_i$ for all $i = 1, \dots, J$, we obtain the capacity inequality

$$\alpha \sum_{j=1}^J p_j \nu_j \leq \min_{1 \leq i \leq J} \left\{ 1 - \frac{\nu_i}{\bar{\Gamma}_i} \right\} \quad (21)$$

For the MMSE receiver we obtain

$$\xi = 1 - \alpha \sum_{j=1}^J p_j \nu'_j,$$

where

$$\nu'_j \triangleq \int_0^\infty \frac{x \nu_j}{1 + x \nu_j} dF_z(x),$$

provided that

$$\alpha \sum_{j=1}^J p_j \nu'_j \leq 1$$

The resulting power-control equation is

$$\Gamma_i = \frac{\nu_i}{1 - \alpha \sum_{j=1}^J p_j \nu'_j}$$

By imposing the input constraint $\Gamma_i \leq \bar{\Gamma}_i$ for all $i = 1, \dots, J$, we obtain the capacity inequality

$$\alpha \sum_{j=1}^J p_j \nu'_j \leq \min_{1 \leq i \leq J} \left\{ 1 - \frac{\nu_i}{\bar{\Gamma}_i} \right\} \quad (22)$$

The effective bandwidth is given by

$$B_i^{(s)} = \nu_i$$

for the SUMF and by

$$B_i^{(m)} = \nu'_j$$

for the MMSE.

Example 2. Assume Rayleigh fading, i.e., $F_z(x) = (1 - \exp(-x))u(x)$. From (9) we have

$$C = e^{1/(\xi\Gamma)} \text{Ei}(1, 1/(\xi\Gamma)) \log_2 e \quad (23)$$

where

$$\text{Ei}(1, x) \triangleq \int_1^\infty \frac{e^{-xt}}{t} dt$$

and $\xi = \xi^{(s)}$ or $\xi = \xi^{(m)}$, depending on the receiver used.

Fig. 3 shows the capacity curves for SUMF and MMSE as a function of E_b/N_0 for different values of α with Rayleigh fading. With the SUMF the capacity is bounded for all α , while with the MMSE it is bounded only for $\alpha \geq 1$ (interference-limited condition). Fig. 4 shows the system capacity regions with SUMF and MMSE receiver in a system with two user groups: the first one has 90% of the users transmitting with a signal-to-noise ratio 10 dB, and the second has the remaining users with a signal-to-noise ratio 13 dB.

4.1 System throughput

Let $\tilde{\xi} = \beta_1/z_1$ and $\Gamma = RE_b/N_0$. Then, (2) and (3) can be rewritten in the form

$$\tilde{\xi} = \frac{R(\tilde{\xi}) \frac{E_b}{N_0}}{1 + \eta \frac{E_b}{N_0} \zeta(\tilde{\xi})} \quad (24)$$

where $R(\tilde{\xi}) = E[\log_2(1 + \tilde{\xi}z)]$ and where

$$\zeta(\tilde{\xi}) = \begin{cases} E[z] & \text{SUMF} \\ E\left[\frac{z}{1+z\tilde{\xi}}\right] & \text{MMSE} \end{cases}$$

By rearranging the terms in (24), we can express directly the throughput η as a function of $\tilde{\xi}$, namely

$$\eta = \frac{1}{\zeta(\tilde{\xi})} \left[\frac{R(\tilde{\xi})}{\tilde{\xi}} - \frac{1}{E_b/N_0} \right]_+ \quad (25)$$

Assuming $E[z] = 1$, we have that

$$\lim_{\tilde{\xi} \rightarrow 0} \eta = \left[\log_2 e - \frac{1}{E_b/N_0} \right]_+ \quad (26)$$

which is positive for $E_b/N_0 > \log 2 = -1.59$ dB. Moreover, from (24) and from the fact that $\eta = \alpha R(\tilde{\xi})$ it is immediate to see that $\tilde{\xi}$ vanishes as $\alpha \rightarrow \infty$. Then, for both the SUMF and the MMSE receivers we have that (26) is also the limit of η for large channel load α (this limit is the same as for a non-fading channel [4]).

The function $\eta(\tilde{\xi})$ defined by (25) is non-increasing for all E_b/N_0 for the SUMF. This implies that the maximum η with SUMF reception is obtained for $\alpha \rightarrow \infty$ and vanishing per-user rate $R(\tilde{\xi}) \rightarrow 0$. With MMSE reception $\eta(\tilde{\xi})$ is non-increasing for $E_b/N_0 \leq (E_b/N_0)_{\text{th}}$, while it has a maximum for positive $\tilde{\xi}$ for $E_b/N_0 > (E_b/N_0)_{\text{th}}$ where $(E_b/N_0)_{\text{th}}$ is a threshold value (the same behavior is noticed in a non-fading channel [4, 14]). Remarkably, for $E_b/N_0 \leq (E_b/N_0)_{\text{th}}$ the system throughput is maximized by $\alpha \rightarrow \infty$. Therefore, there is no point in using a MMSE detector since the same maximum throughput is achieved by the SUMF detector. We can calculate explicitly the value of $(E_b/N_0)_{\text{th}}$ by solving the inequality

$$\left. \frac{\partial \eta(\tilde{\xi})}{\partial \tilde{\xi}} \right|_{\tilde{\xi}=0} \geq 0$$

For all fading distributions with $E[z] = 1$ we obtain $(E_b/N_0)_{\text{th}} = 2 \log 2 = 1.41$ dB (the same value as for a non-fading channel [14]). Fig. 5 shows the qualitative behavior of η as a function of $\tilde{\xi}$ for E_b/N_0 below and above the threshold, for SUMF and MMSE.

Figs. 6 and 7 show both C and η vs. α at $E_b/N_0 = 1$ dB and 6 dB, respectively, for Rayleigh fading and a channel with no fading. The presence of a maximum throughput for finite α when $E_b/N_0 > (E_b/N_0)_{\text{th}}$ for the MMSE receiver is clearly visible.

Fig. 8 shows the system throughput optimized with respect to α as a function of E_b/N_0 , for the SUMF and MMSE receivers, for Rayleigh fading and no fading (analogous results are shown in [9] also for other types of receivers with different power control strategies).

Finally, Fig. 9 shows the behavior of α_{opt} as a function of E_b/N_0 for the MMSE receiver with and without Rayleigh fading. As $E_b/N_0 \rightarrow (E_b/N_0)_{\text{th}}$ from the right, α_{opt} diverges. In the range $\log 2 \leq E_b/N_0 \leq 2 \log 2$, $\alpha_{\text{opt}} \rightarrow \infty$ and MMSE reception is useless for the sake of maximizing the throughput. Notice that α_{opt} exhibits a minimum (this occurs at $E_b/N_0 \approx 10$ and ≈ 18 dB for no fading and Rayleigh fading, respectively). This behavior can be explained by noting that for low E_b/N_0 , the MMSE receiver approaches the SUMF (noise dominates multiple-access interference in this case), and the system throughput is maximum when α is large. For high E_b/N_0 , the MMSE receiver approaches the decorrelating detector [1] (this is the optimal linear receiver in the absence of noise), and system throughput is maximum for $\alpha \rightarrow 1$. Moreover, α must tend to 1 from below, since for large Γ we get $\alpha \simeq 1 - \xi^{(m)}$ with $\xi^{(m)} \leq 1$. Then, by continuity, α must have a minimum for some $E_b/N_0 > 2 \log 2$.

As far as the effect of fading on the system throughput is concerned, Rayleigh fading always decreases throughput with the SUMF while for large α and MMSE detection it provides a modest throughput increase. This can be interpreted as a sort of implicit “load control” operated by fading (see [9]): the fraction of relevant interferers per chip is actually smaller than α because some users experience deep fading. The dimensional crowding problem of the linear MMSE receiver is alleviated by fading, and for large α the benefit of this effect is larger than the degradation due to the fading of the useful signal component.

4.2 Spreading-coding tradeoff

We use the above analysis to dimension a non-asymptotic CDMA system with total bandwidth W , user information bit-rate R_b and transmit power P . The energy per bit is given by $E_b = P/R_b$ and the bandwidth expansion is given by W/R_b . Since P , W and R_b are system constraints and we assume N_0 given, both E_b/N_0 and W/R_b are fixed. The bandwidth expansion should be apportioned between spreading and coding, so that $L/R = W/R_b$, where R is the user coding rate, expressed in information bits per symbol, and L is the spreading factor, expressed in dimensions per symbol. By optimal spreading-coding trade-off we mean to dimension the system so that η is maximum, i.e., select $\alpha = \alpha_{\text{opt}}$ and $R = C(\alpha_{\text{opt}})$. As a consequence, the optimized spreading gain is obtained as $L = (W/R_b)C(\alpha_{\text{opt}})$. For a system based on SUMF, the system throughput is maximized for $\alpha \rightarrow \infty$. This implies a very large number of users, each of which transmitting at a low coding rate. In this case, as it is well-known, the whole bandwidth expansion should be devoted to (low-rate) coding, while devoting a minimum amount of spreading to acquisition and synchronization [13]. On the contrary, for a system equipped with an MMSE receiver, we observe that for $E_b/N_0 > (E_b/N_0)_{\text{th}}$ there is a finite α_{opt} , otherwise α_{opt} is infinite and SUMF is good enough.

Example 3. Consider a system with parameters $W = 4$ MHz, $R_b = 16$ kb/s and $E_b/N_0 = 6$ dB (these figures are inspired by UMTS [6]). From Fig. 7 we see that with MMSE $\alpha_{\text{opt}} \approx 1.3$ and $C(\alpha_{\text{opt}}) \approx 1.2$. This yields $L = 300$ and $K = 390$. A coding rate of 1.2 bit/symbol can be approximated, for example, by binary turbo coding of rate 4/7 concatenated with QPSK, or binary coding of rate 3/7 concatenated with 8PSK, where efficient implementations for binary coding rates 4/7 and 3/7 can be obtained by suitably puncturing mother codes of rate $1/n$ [17]. Practical system values with conventional techniques are $L = 256$ and $K \leq 100$ [6]. Therefore, numbers provided by asymptotic analysis appear quite realistic and, in passing, show the potential benefit of linear interference rejection techniques and powerful channel coding, at least in the case of an isolated cell.

5 Conclusions

We have examined a CDMA system operating on a channel affected by frequency-flat fading. The receiver consists of either a SUMF or an MMSE filter. We have treated separately the cases of slow and fast fading, yielding non-ergodic and ergodic channels, respectively. In the non-ergodic case, we have studied the outage probability and the system outage capacity. In the ergodic case, performance was expressed in terms of system throughput and system capacity. Among our findings, we have proved that in a slow-fading regime the outage probability of the SUMF receiver exhibits an error floor for large SNR and all channel loads, while the MMSE receiver does not whenever $\alpha < 1$. Also, we showed that in a fast-fading regime as $\alpha \rightarrow \infty$ the system throughput with SUMF and MMSE tends to the same limit as for the non-fading channel with the same average SNR. We have also shown that there exists a threshold of E_b/N_0 above which the MMSE receiver does not provide any benefit over the SUMF in terms of throughput maximization. Finally, we have addressed the tradeoffs involved in the allocation of available bandwidth between spreading and coding, and we showed that the asymptotic analysis based on random spreading sequences gives actually meaningful and easy-to-compute results and may serve as a tool to dimension practical finite-size systems.

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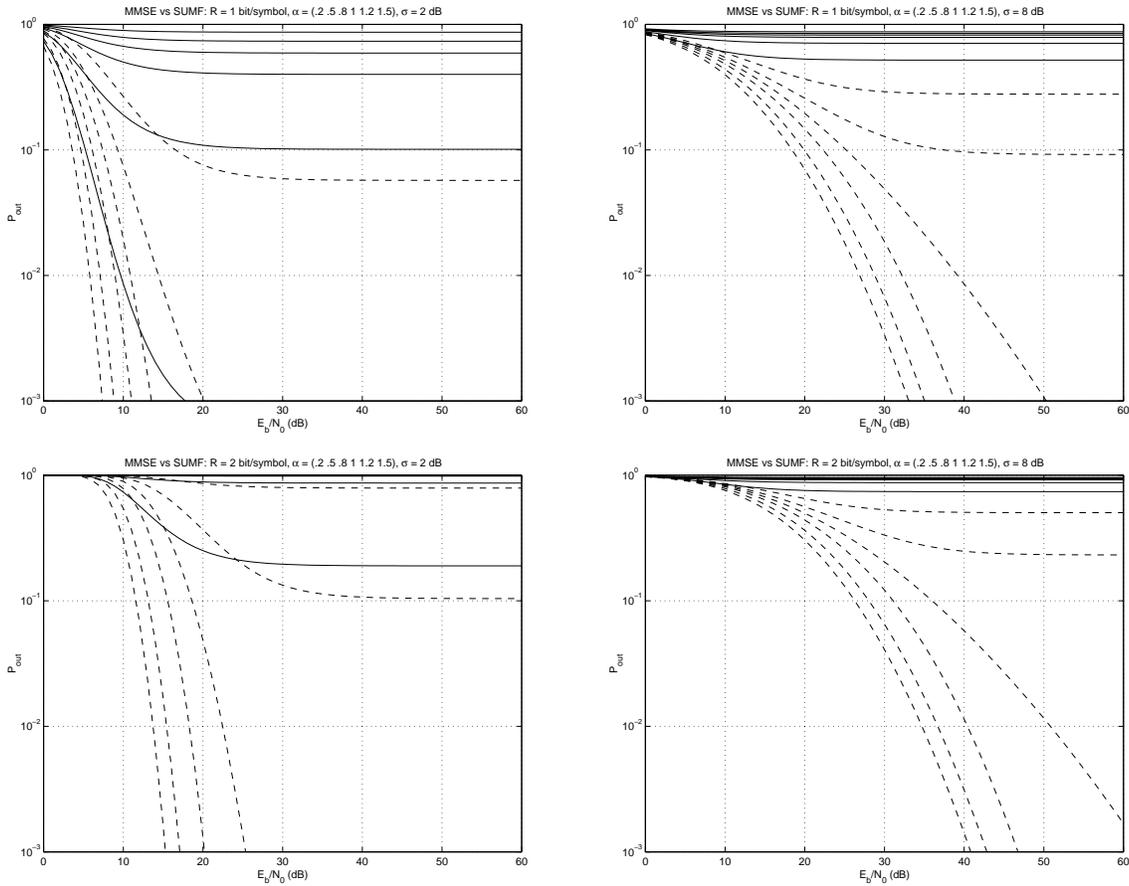


Figure 1: Outage probability with MMSE (dashed) and SUMF (solid) for rates $R = 1$ and 2 bit/symbol, $\sigma = 2$ and 8 dB, and $\alpha = 0.2, 0.5, 0.8, 1, 1.2, 1.5$.

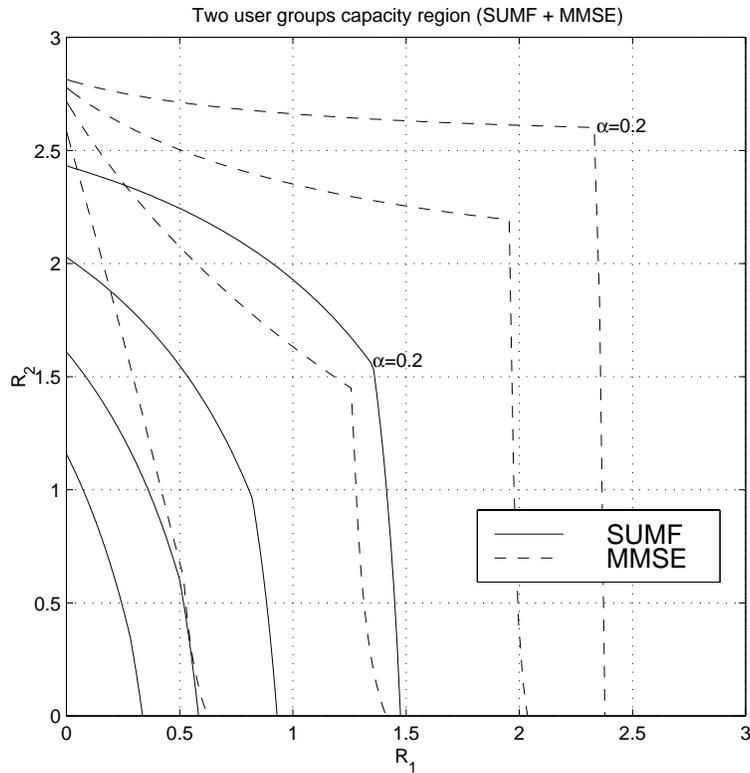


Figure 2: System outage capacity regions with two user groups. The first one has 90% of the users transmitting with an outage probability $P_{\text{out},1} = 0.1$ and a signal-to-noise ratio $\bar{\Gamma}_1 = 10$ dB; the second one has 10% of the users transmitting with an outage probability $P_{\text{out},2} = 0.01$ and a signal-to-noise ratio $\bar{\Gamma}_2 = 13$ dB. The channel statistics are log-normal with log-standard deviation $\sigma = 2$ dB, and $\alpha = 0.2, 0.5, 1,$ and 2 .

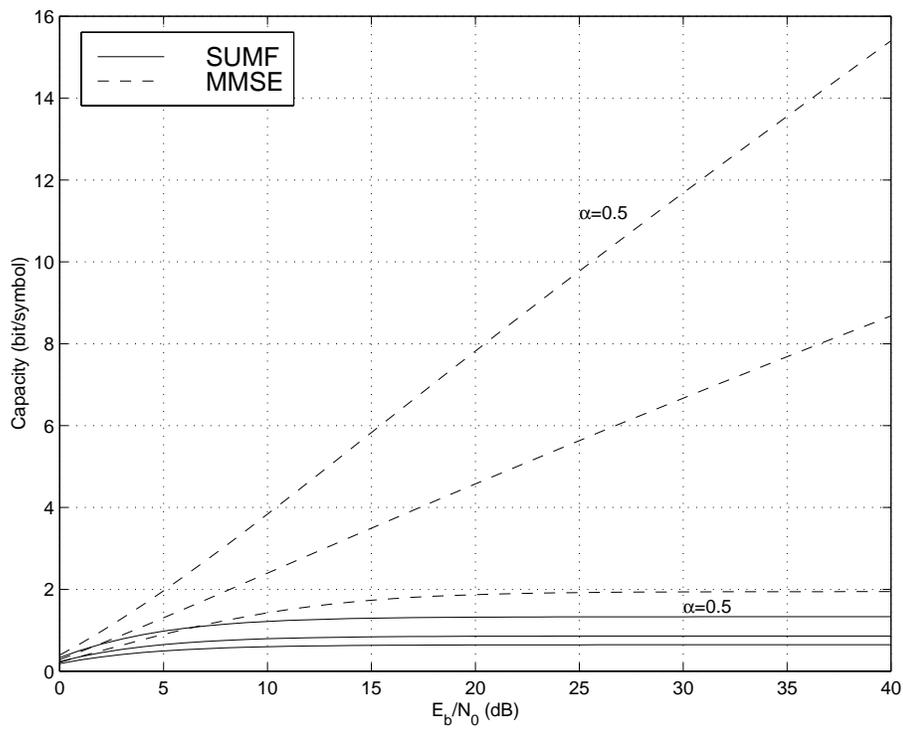


Figure 3: Capacity of MMSE (dashed) and SUMF (solid) (bit/symbol), $\alpha = 0.5, 1, 1.5$.

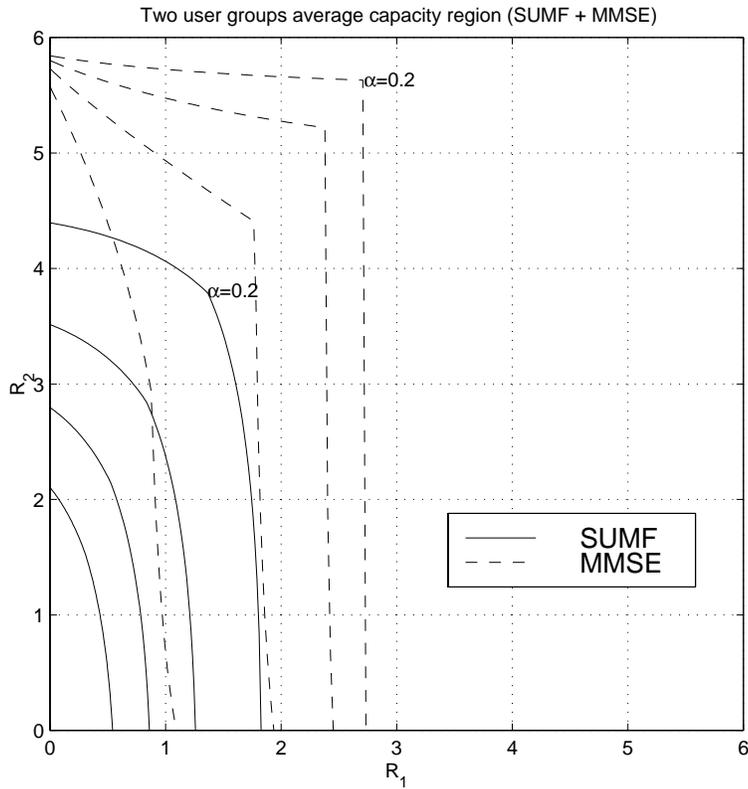


Figure 4: System capacity regions with two user groups. The first one has 90% of the users transmitting with a signal-to-noise ratio 10 dB; the second one has 10% of the users transmitting with a signal-to-noise ratio 13 dB. The channel statistics are Rayleigh, and $\alpha = 0.2, 0.5, 1, \text{ and } 2$.

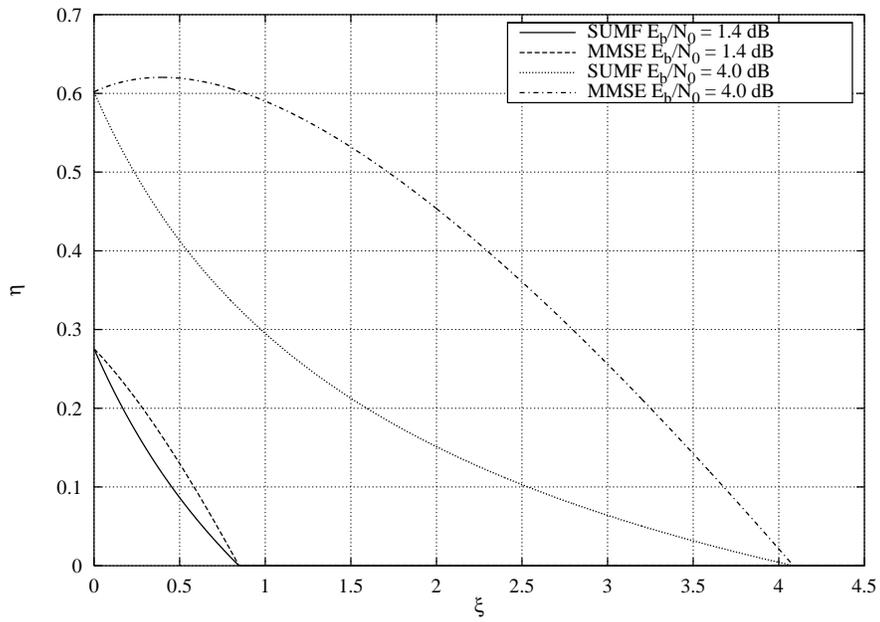


Figure 5: Qualitative behavior of η as a function of $\tilde{\xi}$ for E_b/N_0 below and above the threshold, for SUMF and MMSE.

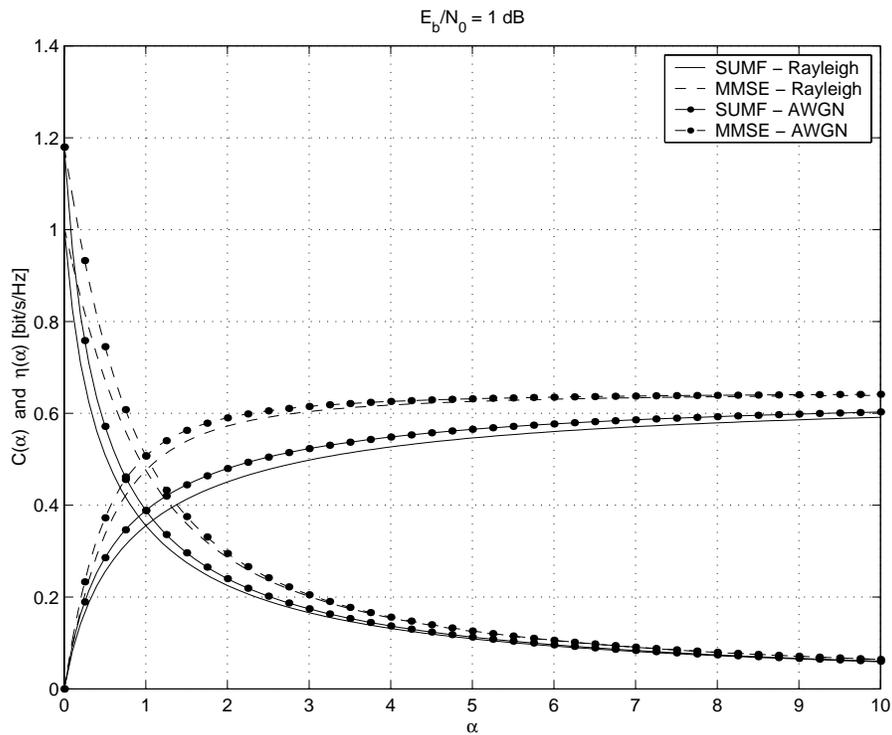


Figure 6: Plot of $C(\alpha)$ — monotonically decreasing curves — and $\eta(\alpha)$ vs. α for the MMSE (dashed) and SUMF (solid) receiver for $E_b/N_0 = 1$ dB.

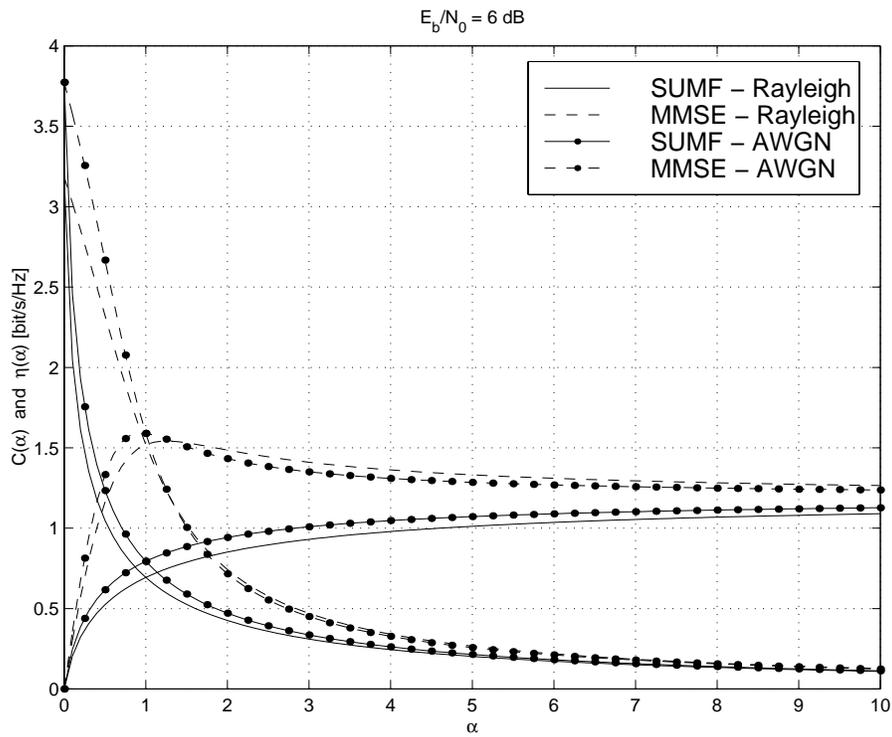


Figure 7: Plot of $C(\alpha)$ — monotonically decreasing curves — and $\eta(\alpha)$ vs. α for the MMSE (dashed) and SUMF (solid) receiver for $E_b/N_0 = 6 \text{ dB}$.

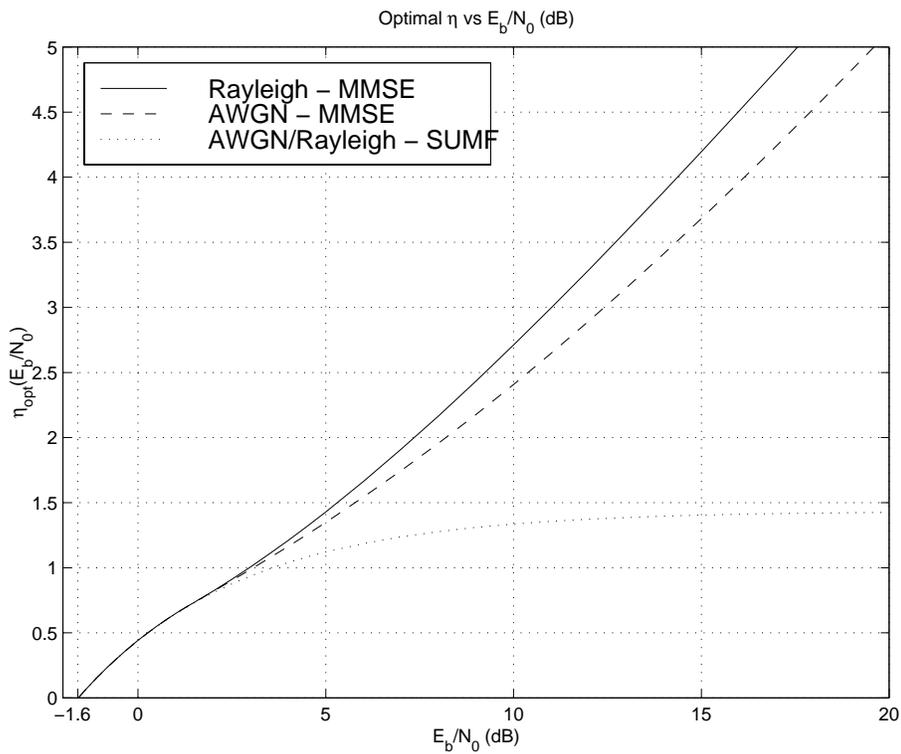


Figure 8: Optimal system throughput η_{opt} versus E_b/N_0 for SUMF and MMSE receivers: AWGN and Rayleigh fading channel.

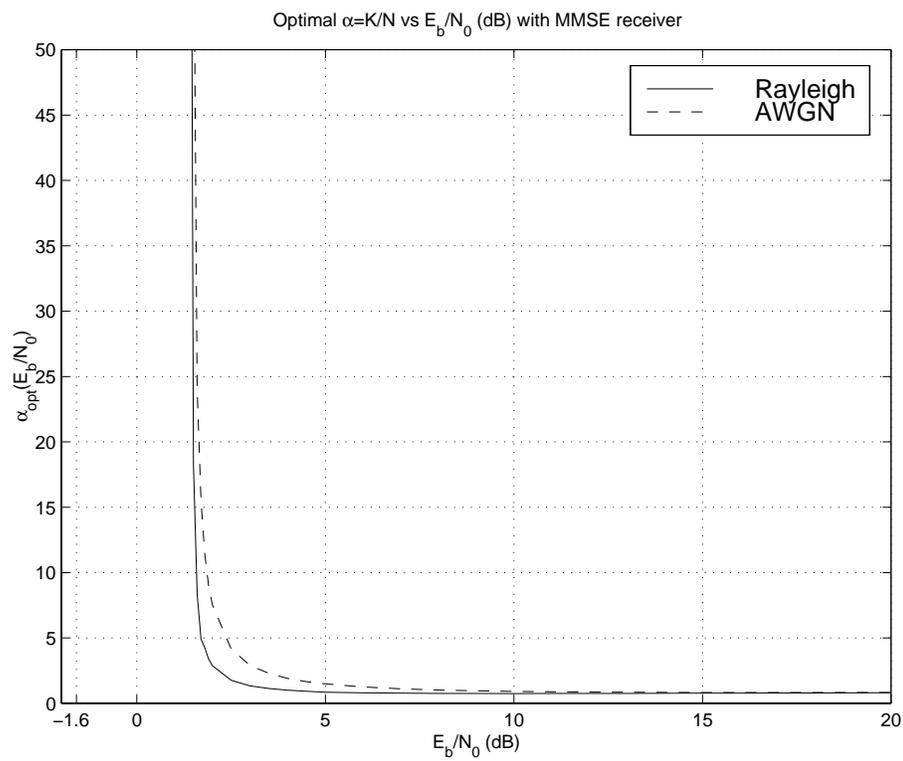


Figure 9: α_{opt} versus E_b/N_0 for the MMSE receiver: AWGN and Rayleigh fading channel.